Compactness, Optimality and Applications

minEquMax 2019, Granada

José Orihuela

Departamento de Matemáticas, Universidad de Murcia, Espinardo (Murcia), Spain

Abstract

One of the most important achievements in optimization in Banach space theory is the James' weak compactness theorem. It says that a weakly closed subset A of a Banach space E is weakly compact if, and only if, every linear form $x^* \in E^*$ attains its supremum over A at some point of A. . We propose a tour around it in the natural framework of variational analysis. Of course, we shall bring related open questions. We will concentrate on recent extensions of James' theorem. Among them we shall study the following one:

Theorem. *Let A be a closed, convex, bounded and not weakly compact subset of a Banach space E. Let us fix a convex and weakly compact subset D of E, a functional $z_0^* \in E^*$ and $\epsilon > 0$. Then there is a linear form $x_0^* \in B_{pw}(z_0^*, \epsilon)$, i.e.*

$$|x_0^*(d) - z_0^*(d)| < \epsilon$$

for all $d \in D$, which does not attain its supremum on A. Moreover, if $z_0^*(A) < 0$ the same can be provided for the former non attaining linear form : $x_0^*(A) < 0$ (one sided James' theorem).

In **the first part** we shall concentrate in the case of Banach spaces with w^*-sequentially compact dual unit ball. We shall present one-sided versions of the well known results by Bishop and Phelps, Simons, Fonf and Lindenstrauss, which play their job and go back to ideas of a joint work with B. Cascales and A. Pérez. (2017)

In **the second part** we shall provide techniques for a proof of Theorem in arbitrary Banach spaces. Our approach comes from Ruiz Galán and Simons and it goes back to the Pryce's undetermined function technique. We will show the strong connection of James' theorem with variational principles and optimization theory. In order to do it, we will study

unbounded versions of the former results. The first case should be the epigraph of a weakly lower semicontinuous function

$$\alpha : E \longrightarrow (-\infty, +\infty],$$

where we shall see that $\partial\alpha(E) = E^*$ if, and only if, the level sets $\{\alpha \leq c\}$ are weakly compact (the Fenchel conjugate α^* should be finite for the "if" implication), which goes back to ideas of Ruiz Galán, Simons, Calvert and Fitzpatrick we have collected in a joint work with M. Ruiz Galán (2012). Moors deserves special mention here since he has recently obtained a closely related variational principle too.

In **the third part** we shall present some new applications. We will see that reflexive spaces are the natural frame to develop variational analysis and we will show a robust representation theorem for a *risk measure* $\rho : \mathbb{L}^\infty \longrightarrow \mathbb{R}$ in natural dual pairs appearing in financial mathematics, both applications are based in a joint work with M. Ruiz Galán. We shall study the Mackey topology $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$ and a new characterization for risk measures ρ verifying the Lebesgue dominated convergence theorem, as the expectated value does. The proof of Theorem together with these applications have been obtained in joint work with F. Delbaen and T. Pennanen (preprint 2018). We shall finish our programme with $\sigma(E^*, E)$ versions of the discussed results. Indeed, we shall look for conditions that provide $\sigma(E^*, E)$-closedness of norm closed convex (not necessarely bounded) subsets of a dual Banach space E^*. One-sided versions of classical Godefroy' results will be presented and new applications considered. Some of them comes from the same joint work with B. Cascales and A. Pérez (2017).